Space and spaces

Graeme Segal

The idea of space is central in the way we think. We organize our perceptions in physical space, we think of time as a one-dimensional continuum, and we carry spatial notions over to any number of our conceptual constructs. Nevertheless, in some sense space is 'our' technology, wonderfully evolved for dealing with our experience of the physical world. It is probably only an approximation to reality.

Long ago I happened to be looking at an account of quantum field theory written 'for the general reader' by Freeman Dyson, one of my favorite writers about science. I came upon a passage where he said that there are 'well-understood mathematical reasons' why when we quantize a wave-like physical system such as the electromagnetic field the result is best described in terms of particles. "Unfortunately," he added, "the reason for this cannot be explained in non-mathematical terms." I felt rather let down, for I don't like the idea of mathematics as an arcane mystery where even the basic ideas can only be explained to initiates. I also wondered exactly which well-understood mathematical reasons Dyson had in mind. I have thought about the question a lot since then, but haven't come up with anything that would be of much use to non-mathematicians; I leave it as a challenge. It seems to me a challenge even to give a clear mathematical account. I suppose that Dyson was thinking of the traditional statement that a free field can be regarded as a system of independent harmonic oscillators — which mystified me so much when I first encountered it in physics lectures as an undergraduate — but I believe that to understand the essential point involves thinking carefully about how we use the concept of space. That is the aim of this talk.

One of the triumphs of mathematics is the creation of the real numbers as a model of a one-dimensional continuum. It perfectly encodes all our intuitions of how such a thing should be[1]. Another great achievement — if

[1]Once when I was young and innocent I tried to explain to an eminent philosopher

not quite on the same level — is the concept of a topological space, which captures a more general intuitive idea. Primarily, a space is a *set* with a notion of *proximity*, which tells us when a point of the the space is moving continuously as a function of time, and tells us when a real-valued function on the space is continuous. Beyond that, it has the feature that different regions in the space can in some sense be studied independently. The formal definition of a topological space does not involve the real numbers, but, as you see, my account of the intuition very much does.

But spaces also have a non-local aspect. The crudest way to classify topological spaces is by their homotopy-type. If we start with the category of topological spaces and identify maps when they are homotopic — when one can be deformed continuously to another — then we get the *homotopy category*. In this category homotopy-equivalent spaces become isomorphic, so, effectively, the homotopy category is very much smaller than the category of topological spaces. It is the receptacle for the *non-local* properties of a space: it records how the space is connected-up globally — what we still need to know when we know everything about the local structure near every point of the space.

We have very powerful intuitions about homotopy-types, at least in low dimensions. No-one ever doubted that a real polynomial equation of odd degree has a real root, long before there was any framework to prove the intermediate-value theorem. In the same way, the winding-number proof that the complex numbers are algebraically closed — the two-dimensional analogue of the intermediate-value theorem — carries immediate conviction[2].

The non-local nature of the homotopy category gives it a life of its own. Though in principle its objects are sets, we often come upon well-defined objects in the homotopy category for which there is no specially singled-out topological space which represents them. The London tube map is obviously well-defined up to homotopy, but what, exactly, is the underlying set? There are many more extreme examples: in mathematical logic and com-

in my Oxford college why mathematicans are no longer puzzled about how Achilles can overtake a tortoise, and the resulting explosion has made me wary of transgressing cultural boundaries.

[2]Our intuition is of course patchy and unreliable: one is rightly certain that if one puts a disc of cloth down on a smaller circular plate so that the edge of the cloth runs twice around the boundary of the plate then the cloth must cover the plate completely, but it is less immediately evident that performing the act with ordinary cloth in ordinary space is impossible.

puter science it can be useful to think of a 'proof' as a 'path' — defined up to homotopy — from one proposition to another. Evidently these paths are maps from an interval of the real numbers into a definite topological space.

One place where the independent life of the homotopy category is important is in algebraic geometry. An algebraic variety over the complex numbers is a space, but not just any space: when a space is defined by algebraic equations its homotopy-type is very strongly constrained, and has an elaborate additional structure. For the moment I'd just like to point out that the homotopy-type of a variety plays a mysterious role in unexpected contexts. I first became interested in mathematics through physics, and I was incurious about number-theoretical questions like Fermat's last theorem. I felt more interested when the theorem was restated to me as the fact that the real plane curve with equation $x^n + y^n = 1$ somehow gets from (1,0) to (0,1) without going through any points with rational coefficients. But I was gripped when I heard of Mordell's conjecture, proved by Faltings in 1983, which gives a criterion for *any* plane curve with a rational equation to have at most *finitely* many rational points. The criterion is in terms of the homotopy-type of the surface formed by the *complex* points of the curve: if the surface has genus greater than 1, i.e. has more than one 'handle', then there are at most finitely many rational points. What can the global topology of the complex variety — or even of the subspace of points with real coordinates — have to do with the rational points? An unromantic answer is that a property of the algebraic equations happens to control both; but the homotopy-type of the surface leaps out at one, while the genus is far from being a salient feature of the algebraic description of the curve.

More magical still is the case of algebraic varieties defined over a finite field by equations with integer coefficients. According to the Weil conjectures, proved by Grothendieck and Deligne, the number of points of such a variety is again related to the homotopy type of the associated *complex* variety. The essential idea of the Weil conjectures is that an algebraic variety over a finite field, though it is not in any ordinary sense a topological space, nevertheless has a homotopy-type which is defined by its algebraic structure. Grothendieck's construction of this homotopy-type began from the observation that to know the homotopy-type of an ordinary space one does not need to know the space as a point-set: all one needs to know is the way its contractible open subsets are fitted together combinatorially. There is a simple construction — making essential use of the real numbers — which associates a space $|\mathcal{C}|$, and hence a homotopy-type, to any *category* $\mathcal{C}$. The partially-ordered set $\mathcal{U}_X$ of contractible open subsets of a space X is a category, with just the inclusions as morphisms. Surprisingly, the homotopy-type of $|\mathcal{U}_X|$ is precisely the homotopy-type of the space X. What really makes the homotopy type of a category significant is that it depends on the category only up to equivalence of categories: the category of "all" finite sets and bijections has the same homotopy type as the countable subcategory whose objects are just the sequence of particular sets $\{1, 2, \ldots, n\}$ for $n \geq 0$, and whose morphisms are the symmetric groups. Grothendieck defined a category of 'generalized open subsets' for an arbitrary algebraic variety, and (oversimplifying slightly) the homotopy-type of the variety is defined as that of this category.

When we prove the fundamental theorem of algebra by homotopy theory we are applying our intuitions to our best model of physical space. But often we use the language of spaces and geometry more like an analogy. We say, for instance, that a module is projective if it is 'locally free', or speak of an 'infinitesimal deformation' of an algebraic variety over an arbitrary field; in these cases there is no genuine space around[3] of the sort to which our intuitions apply. These analogies are nonetheless very powerful tools. It is something of a surprise, therefore, that the homotopy-types we assign to algebraic varieties over arbitrary fields *are* those of genuine spaces, even though the points of the spaces are not those of the variety. Over the last half-century the assignment of homotopy-types to groups, rings, and all manner of other algebraic objects, has become steadily more pervasive in mathematics, since its beginnings when it was called 'homological algebra'. I am as far as possible from being a mathematical logician, but I am intrigued that when I began research fifty years ago one of the avant-garde ideas in logic was that categories and not sets were the right starting point for the foundations of mathematics, while nowadays I hear about Voevodsky's 'univalent foundations' project — a theory of 'types' — which bases the foundations on homotopy theory.

In the same half-century the subject called 'noncommutative geometry' has sprung up. It begins from the rough correspondence — contravariant —

[3] We can, of course, define the Zariski topology on the set of points of an algebraic variety over any field. This is a topological space of a sort; but it is a weird space from the point of view of our intuitions, and when I use it I feel I am employing my powers of analogy rather than my spatial intuition.

between the category of topological spaces and the category of commutative algebras over $\mathbb{C}$. In one direction, we associate to a space X the algebra $C(X)$ of continuous complex-valued functions on X, and, in the other, to a commutative algebra $\mathcal{A}$ we associate the space $\mathrm{Spec}(\mathcal{A})$ of algebra homomorphisms $\mathcal{A} \to \mathbb{C}$, or, equivalently, the space of irreducible $\mathcal{A}$-modules.

Why might a mathematician want to extend this correspondence to include noncommutative algebras? One kind of reason is that the properties of a commutative ring — even if one's interest in it is purely algebraic —- are unquestionably illuminated by thinking in terms of the space it defines, and one can aim for similar illumination about noncommutative rings.

A quite different *kind* of reason is that we encounter mathematical objects which we feel intuitively are 'spaces', but whose space-like properties cannot be captured by the usual concept of a topological space. Standard examples are the space of leaves of a foliated manifold when each leaf is dense in the manifold, and the space of orbits of a group which acts ergodically on a space — e.g. the group of integers $\mathbb{Z}$ acting on the circle $\mathbb{T}$ by an irrational rotation. These badly-behaved quotients of ordinary spaces are the examples which Connes uses to motivate the study of noncommutative geometry, for, even though every scalar-valued continuous function of the leaf or orbit is necesarily constant, nevertheless there are non-constant *operator-valued* functions, and the leaves or orbits precisely parametrize the irreducible representations of a noncommutative algebra $\mathcal{A}$ naturally associated[4] to the foliation or group action. It is useful to think of $\mathcal{A}$ as playing the role of the functions on the space. Another surprise, in these quotient-space examples, is that — as we shall see — there is an ordinary homotopy-type which is naturally associated to the noncommutative algebra.

These examples may seem rather pathological, but the idea can be seen in much simpler situations. The prime examples of quotient spaces are spaces

[4]When a discrete group Γ acts on a compact Hausdorff space X the natural algebra $\mathcal{A}$ is the *twisted group-algebra* $C(X)[\Gamma]$. If Γ acts freely and the quotient-space X/Γ is Hausdorff then $\mathcal{A}$ has the same category of modules as the commutative algebra $C(X/\Gamma)$. The representation of $\mathcal{A}$ associated to an orbit ω of Γ in X is on the Hilbert space $\ell^2(\omega)$, on which Γ acts by translation, and functions $f \in C(X)$ act by multiplication operators, regarding ω as a subset of X. A potentially confusing point, if one wants to think of elements of $\mathcal{A}$ as operator-valued functions on the space of orbits, is that the space $\ell^2(\omega)$ on which the operator acts is changing with the orbit: that is essential, for any operator in a *fixed* space which depended continuously on the orbit would have to be constant for the same reason as a scalar-valued function.

of *isomorphism classes* of objects of various kinds — objects of topological categories, i.e. categories whose sets of objects and of morphisms have a topology[5]. Spaces of isomorphism classes are not quite ordinary spaces. Let us think of the space of dumbbells of weight 1. A dumbbell is described by a pair (a, b) of positive numbers such that $a + b = 1$, but we have the isomorphism $(a, b) \cong (b, a)$. Thus the space of dumbbells is the quotient of the open interval $(0, 1)$ by the relation $a \sim 1 - a$, and it looks as if that is the same as the half-open interval $(0, 1/2]$. But the situation is more complicated. Is the space simply-connected? Consider the obvious closed path of dumbbells from $(1/3, 2/3)$ to $(2/3, 1/3)$. It seems one can contract it to the constant path $(1/2, 1/2)$. But when one does that one doesn't get a constant dumbbell: one gets the boundary of a Möbius band. The "true" fundamental group of the space is of order 2. Noncommutative geometry handles this situation very satisfactorily. Any topological category defines a noncommutative space[6], and equivalent topological categories define the same noncommutative space[7]

Why do we care about spaces of isomorphism classes? One of the striking discoveries of twentieth-century physics was that *all* of the state-spaces of fundamental physics are of this kind: in the language of physics, they are *gauge theories*. In the nineteenth century it was believed that a state of the electromagnetic field, in the absence of particles, was described by its field-strength, a tensor field on space-time which satisfies Maxwell's equations. But in the twentieth century it was realized that a more subtle idea is needed: an electromagnetic field is not a function on space-time but rather an *object*, in fact a pair (L, A) consisting of a complex hermitian line-bundle L on space-time equipped with a connection A, up to isomorphism of such pairs.

[5] When one has an equivalence relation on a space X one can think of the points of X as the objects of a category, and an equivalence $x \sim x'$ as a morphism from x to x', so that the set of morphisms of the category is a subspace of $X \times X$. Just like a discrete category, a topological category $\mathcal{C}$ defines a space $|\mathcal{C}|$, and hence a homotopy-type, and this is the homotopy type of the noncommutative quotient-space which was just mentioned

[6] via a straightforward generalization of the twisted group-algebra which described the orbits of a group-action

[7] The topological category describing the action of the cyclic group $\mathbb{Z}$ on the circle $\mathbb{T}$ by multiplication by $e^{i\theta}$ is a subcategory of that describing the foliation of the torus $\mathbb{T} \times \mathbb{T}$ by lines of slope θ. The inclusion of topological categories is an equivalence, so the noncommutative spaces are the same. This lets us see why the homotopy-type of the quotient of the circle by an irrational rotation is that of a torus.

The classical field strength is the curvature of the connection, and it must satisfy Maxwell's equations. If we want to include electrons, then a state is an isomorphism class of triples (E, A, ψ), where the line-bundle L has been replaced by a complex spinor-bundle E, still with a connection A, but with a section ψ in addition[8]. Similarly, in general relativity it is crucial that a state of the gravitational field is not a metric tensor on a fixed space-time which satisfies Einstein's equations, but rather an isomorphism class of pseudo-Riemannian manifolds.

Let us think in more general terms about the uses of space in physics. Obviously there are the space and time in which we live. But in classical (nonrelativistic) physics, whenever we single out a part of the world we encounter a new space: the space Y of *states* of the system we are considering. Usually the state consists of the instantaneous configuration, which is a point of a *configuration manifold* X, together with its instantaneous rate of change. Thus Y is the tangent bundle TX of X. Physics tells us that Y is a Poisson manifold (i.e. it comes with a bracket operation

$$\{\ ,\ \}:\ \mathrm{C}^\infty(Y) \ \times\ \mathrm{C}^\infty(Y) \ \longrightarrow\ \mathrm{C}^\infty(Y)$$

on the vector space $\mathrm{C}^\infty(Y)$ of smooth functions $Y \to \mathbb{R}$ which makes $\mathrm{C}^\infty(Y)$ a Lie algebra), and there is a function $H : Y \to \mathbb{R}$ called the *energy* or *Hamiltonian* such that the time-evolution of the states in Y is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} f \ = \ \{H, f\},$$

where f is any element of $\mathrm{C}^\infty(Y)$.

Passing from classical to quantum physics forces the state-spaces Y to be something more general than ordinary topological spaces. If we want to study quantum gravity then we probably need to rethink space-time itself, but we shall not go down that road: this talk is specifically directed at quantum field theory. Classically, space-time[9] $M \times \mathbb{R}$ is given to us, and the state-space $Y = Y_M$ of the world is TX_M, where a point of the configuration-space X_M consists of a finite subset σ of M — the positions of particles — together with some "fields", which are smooth functions[10] defined only in the

[8]The triple must satisfy the coupled Maxwell and Dirac equations.

[9]For simplicity, in this talk I shall assume that space-time is a the product of a space manifold M with the time-axis $\mathbb{R}$.

[10]More accurately, sections of some bundle on M.

complement $M \setminus \sigma$ of the particles. The worst complication — ultimately fatal for classical physics — lies in the difficulty of prescribing how the fields behave in the neighbourhood of the particles.

In one way quantum field theory is enormously simpler than its classical counterpart, for there are no particles: the manifold of configurations X_M is simply a space of smooth fields on M. In exchange for this simplification, the state-space Y_M can no longer be interpreted as an ordinary space: it is an object of noncommutative geometry, and that is why when we look at the world we sometimes see particles and sometimes see fields. But before coming to that I must emphasize a philosophical point. A classical dynamical system (Y, H) consisting of a Poisson manifold Y and a Hamiltonian function $H : Y \to \mathbb{R}$ is not recognizable as a description of the world — if Y is 60-dimensional it might equally well describe ten particles moving in $\mathbb{R}^3$ or a single particle moving in $\mathbb{R}^{30}$. The physics of the situation lies in the prescription or *functor* which tells us how $Y = Y_M$ is constructed from the physical space-manifold M, in other words, it lies in the prescription telling us that the world consists of particles and fields. A quantum system is a noncommutative analogue of a pair (Y, H) consisting of a Poisson manifold and a Hamiltonian, and it has a similar lack of relation to what we see. Quantum field theory, on the other hand, is the description of the functor $M \mapsto Y_M$ which assigns to each space M a quantum system Y_M. It *does* aim to describe the world.

I believe quantum theory forces us to accept that the truest description of the physical world is in terms of algebraic structures. Nonrelativistically — and our intuitions are certainly nonrelativistic — the structure which nature seems to provide is a noncommutative topological[11] $*$-algebra $\mathcal{A}$ of observables. We form our picture of the world, however, by recognizing the noncommutative algebra as a small deformation[12] $\mathcal{A} = \mathcal{A}_h$ of a commutative algebra $\mathcal{A}_0$. More precisely, out of the vast torrent of observables which the world presents to us we select a subalgebra $\mathcal{A}_h$ which we can recognize as a very small deformation of a commutative algebra. Then we can identify $\mathcal{A}_h$ with $\mathcal{A}_0$ as a vector space, and can define a Poisson bracket on $\mathcal{A}_0$ as the

[11] To an algebraist we are here letting in topology by the back door; but quantum theory does require us to know when *observables* are close — otherwise a self-adjoint operator would not have a spectrum. I shall return to this point later.

[12] The meaning of this needs care. In the typical examples the isomorphism-class of $\mathcal{A}_h$ is independent of h if $h \neq 0$, but jumps when $h = 0$.

departure of $\mathcal{A}_h$ from commutativity, to leading order in h, i.e.

$$\{f, g\} \;=\; (fg - gf)/\mathrm{i}h,$$

where the product on the right is that of $\mathcal{A}_h$. This gives us a space of states Y defined by the commutative algebra $\mathcal{A}_0$, a Poisson structure on Y, and a time-evolution which arises from the noncommutativity and a Hamiltonian element $H \in \mathcal{A}_h$.

Let us quickly review some of the ways in which noncommutative geometry is different.

Passing from commutative to noncommutative geometry we lose not only the idea of points — for the "points" of a noncommutative algebra are its irreducible representations, which do not usually form a reasonable space — but also, in general, the idea that different parts of the space can be studied independently. It might at first seem that we have lost geometry altogether. That is not the case, but it is true that the category of noncommutative spaces has some features resembling the homotopy category more than the category of spaces. Indeed much of noncommutative geometry is about the homotopy-types of the objects — it is focussed on algebraic-topological ideas like cyclic homology and K-theory.

Another 'ungeometrical' feature of noncommutative geometry arises because in the usual interpretation the actual geometric object is not the noncommutative ring or algebra $\mathcal{A}$ itself, but rather the *category* of left $\mathcal{A}$-modules. Thus the algebra $\mathrm{Mat}_n(\mathbb{C})$ of $n \times n$ complex matrices is just a point in the eyes of noncommutative geometry, whatever the value of n, and for any algebra $\mathcal{A}$ the algebra $\mathrm{Mat}_n(\mathcal{A})$ defines the same noncommutative space as $\mathcal{A}$. Because the modules form an additive category this means we can *add* morphisms of noncommutative spaces. In geometrical language, we automatically include multi-valued maps along with ordinary maps, and — in my view — this is the fundamental reason that quantum field theory deals with assemblies of identical particles rather than single particles. There is a best-possible way of adjoining maps to the homotopy category so as to make it additive. The result is the *stable homotopy* category, about which I shall say a little more presently; and it does indeed turn out that noncommutative spaces have stable homotopy types rather than the usual sort.

Yet another perspective on the passage from commutative to noncommutative geometry focusses on inadequacies of the standard notion of a topological space. Topological spaces fall short of our needs in two dual respects. I have already mentioned the problem of 'bad' quotient constructions, but there is a dual inadequacy in dealing with 'bad' subspaces.

The prime examples of "bad subspaces" are spaces of solutions of systems of equations. The simplest illustration is a root of multiplicity $m > 1$ of a polynomial equation in one variable, where a single point needs somehow to remember that it is potentially m distinct points. Spaces where the points have this kind of additional structure can be handled very simply in terms of algebra, and in this case only commutative algebra is needed. If $f \in \mathbb{C}[x]$ is a polynomial with distinct roots then the quotient ring $\mathcal{A} = \mathbb{C}[x]/(f)$ is the usual algebra of functions on the set of roots of f. If the roots are not distinct the ring has nilpotent elements and is no longer the algebra of functions on a set, but, following Grothendieck, one can still think of it as defining a space in which the points have multiplicities: it is an 'infinitesimal thickening' of the actual set of roots. (The general situation — nowadays called a *Kuranishi structure* — is the zero-set of a smooth section of an infinite-dimensional vector bundle on an infinite-dimensional smooth manifold, where the derivative of the section is everywhere Fredholm.)

Whereas the "bad quotient spaces" which were mentioned above do have well-defined homotopy types in the usual sense, "bad subspaces" have only *stable* homotopy types, in the sense that one can at best say what the homotopy type becomes after iterated suspension[13]. To understand why this is so, let us think of the "overdetermined" case when X, though non-empty, is defined as $f^{-1}(0)$, where f is a proper smooth immersion $f : U \to \mathbb{R}^n$ of a manifold U of dimension $m < n$. Then, morally, X has the negative dimension $m - n$, for if f is perturbed a little it will disappear completely, and one has to probe with an $(n - m)$-dimensional family to find it.

Both deficiencies of the notion of topological space are relevant in quantum theory. I have said enough about quotients. We meet noncommutative geometry much more directly when we look for the origin of the Poisson bracket of functions on the classical state-space. From the quantum perspective it is the residual vestige of noncommutativity, but in classical physics it arises because the evolution of the state is governed by a *variational principle* — the "principle of least action". In this picture the classical trajectories sit as a subspace — the solutions of the Euler-Lagrange equation for the action

[13]The n-fold suspension of a compact space X is the one-point compactification of $X \times \mathbb{R}^n$.

functional — in the much larger space of trajectories which need not satisfy the equations of motion. In this talk I cannot hope to give an account of the relation between the noncommutativity and the variational principle, so I shall just say that one way to pass from classical to quantum physics is to integrate over all trajectories in the classical configuration space, weighted according to their action (so that the trajectories obeying the classical equations of motion provide the leading contribution). From that viewpoint the semi-classical approximation is the study of an infinitesimal thickening of the classical state-space.

As I have said, quantum theory suggests looking at a noncommutative space in terms of the commutative space of which it is a small deformation. This is a somewhat different picture from the usual one of noncommutative geometry, but it is helpful for understanding the wave-particle duality of quantum field theory. I shall spend the remainder of this talk describing how noncommutative geometry can make a space of fields look like an assembly of particles.

The simplest kind of field on a compact Riemannian manifold M is described by a smooth real-valued function on M, and so the configuration space X_M is the infinite-dimensional vector space $\mathrm{C}^\infty(M)$, and the classical state-space is its tangent bundle

$$Y_M \;=\; \mathrm{C}^\infty(M) \oplus \mathrm{C}^\infty(M).$$

We take the energy of a state $(\phi, \dot{\phi}) \in Y_M$ to be given by the quadratic form

$$\frac{1}{2}\int_M \{\dot{\phi}(x)^2 + \|\mathrm{d}\phi(x)\|^2 + m^2\phi(x)^2\}\mathrm{d}x.$$

What we want to explain is how noncommutativity makes the algebra of functions on the vector space Y_M look like the functions on the tangent bundle of the configuration space

$$X_M^{\mathrm{part}} \;=\; \coprod_{n\geq 0}(M^n/\mathrm{Symm}_n)$$

of an indefinite number of indistinguishable particles moving in M.

The first point is that quantum field theory tells us that the appropriate commutative algebra $\mathcal{A}_0$ of smooth functions on the infinite-dimensional space Y_M is more subtle than the schematic description $C^\infty(Y_M)$ suggests. It does *not* contain the smooth function $(\phi, \dot\phi) \mapsto \phi(x)$ given by evaluation at a point x of M. Traditionally this is expressed by saying that $\phi(x)$ is 'fluctuating too fast' to have an 'expected value'. On the other hand it does contain the 'smeared-out' function

$$(\phi, \dot\phi) \quad \mapsto \quad \phi_f \;=\; \int_M f(x)\phi(x)\,\mathrm{d}x,$$

for every smooth test-function f with compact support on M. This — together with the corresponding smearings of $\dot\phi$ — gives us a linear map $Y \to \mathcal{A}_0$ which extends to a map of algebras

$$S(Y) \;\to\; \mathcal{A}_0$$

from the symmetric algebra of Y.

This map of algebras is a dense embedding. Nevertheless, there are still many different topologies in which one might complete the symmetric algebra, and quantum field theory picks out a topological algebra $\mathcal{A}_0$ of functions on Y_M which is intimately related[14] to the geometry of the manifold M.

Let us now see how the existence of a noncommutative deformation of the algebra $\mathcal{A}_0$ completely changes the geometric picture.

The crucial example which gives the idea is the abstract polynomial algebra $\mathbb{C}[a]$ in a single variable. This is a dense subalgebra of the algebra $C^\infty(\mathbb{R})$ of functions on the real line, and also of the algebra $C(\mathbb{N})$ of functions[15] on the discrete space $\mathbb{N}$ of positive integers. But $C(\mathbb{N})$ has a quite different algebraic structure from either $\mathbb{C}[a]$ or $C^\infty(\mathbb{R})$, being generated by idempotents e_k for $k \geq 0$ such that $e_k e_m = 0$ when $k \neq m$.

Whether the algebra $\mathbb{C}[a]$ should be regarded as consisting of functions on the line $\mathbb{R}$ or on a closed subset Σ of it — i.e. determining the *spectrum*

[14]For example, one might guess that the appropriate completion of the symmetric square $S^2(C^\infty(M))$ should be the space of symmetric functions in $C^\infty(M \times M)$, but in fact it also contains smooth delta-functions along the diagonal in $M \times M$, i.e. it contains the functions obtained by smearing $\phi(x)^2$. More generally, the QFT completion of $S(Y)$ contains all functions of $(\phi, \dot\phi)$ obtained by smearing any differential polynomial in $\phi(x)$ and $\dot\phi(x)$, such as the Hamiltonian density itself.

[15]For any closed $\Sigma \subset \mathbb{R}$ we give $C(\Sigma)$ the topology of uniform convergence on compact subsets.

Σ of the operator a — is prescribed by the *topology* on the algebra, which is an essential part of the quantum-mechanical description. The question is for which points $\lambda \in \mathbb{R}$ the evaluation-map $f \mapsto f(\lambda)$ is continuous[16]. There is a simple algebraic mechanism which can force on $\mathbb{C}[a]$ the topology induced from $\mathrm{C}(\mathbb{N})$ rather than from, say, $\mathrm{C}^\infty(\mathbb{R})$. Suppose that $\mathbb{C}[a]$ arises as a subalgebra of the noncommutative $*$-algebra $\mathbb{C}\langle b, b^*\rangle$ generated by an element b such that $b^*b - bb^* = 1$, and that a is the self-adjoint element bb^*.

Theorem *If a $*$-action of $\mathbb{C}[a]$ on a Hilbert space $\mathcal{H}$ extends to an action of $\mathbb{C}\langle b, b^*\rangle$, then the action extends canonically from $\mathbb{C}[a]$ to $\mathrm{C}(\mathbb{N})$.*

This theorem is a version of the Stone-von Neumann theorem on the uniqueness of the irreducible representation of the Heisenberg algebra. The way it is relevant at the moment is that if we are presented with a system represented by the algebra $\mathcal{A} = \mathbb{C}\langle b, b^*\rangle$ with the Hamiltonian element a multiple of a then we may choose to see it as a small deformation of the commutative algebra $\mathbb{C}[b, b^*]$ — and then we can see a particle oscillating on a line — but (depending on the scales of the observable elements) the whole algebra $\mathcal{A}$ may not be sufficiently commutative, and we may model the system just by the subalgebra $\mathbb{C}[a]$, in which case we see a stationary system with just a discrete sequence of states. The theorem has a far-reaching (but not much harder to prove) generalization which explains why we see particles and not waves when we look at the quantum algebra of the state-space of fields on a manifold M.

Theorem *If $\mathcal{A}_h$ is the standard Heisenberg deformation of the commutative algebra*[17] *$\mathcal{A}_0$ of functions on the symplectic vector space*

$$Y_M \;=\; \mathrm{C}^\infty(M) \oplus \mathrm{C}^\infty(M),$$

then there are commuting elements $a_f \in \mathcal{A}_h$ for $f \in \mathrm{C}^\infty(M)$ which generate an algebra isomorphic to $\mathrm{C}^\infty(X_M^{\mathrm{part}})$, where

$$X_M^{\mathrm{part}} \;=\; \coprod_{n\geq 0} (M^n/\mathrm{Symm}_n).$$

[16] Operator algebraists like to hide the topology in algebra by encoding it in the way the algebra is completed, but I feel that obscures the real issue.

[17] This is a little disingenuous, in that I am assuming that the topology of $\mathcal{A}_h$ is sufficiently fine to allow us to form the operators a_f. A more satisfactory formulation of the theorem would need a fuller account of quantum field theory.

Here a_f corresponds to the symmetric function $(x_1, \ldots, x_n) \mapsto \sum f(x_i)$ on $M \times \ldots \times M$, though as a function on Y_M it is obtained by smearing the quadratic function $a(x)$ which takes $(\phi, \dot{\phi})$ to

$$\frac{1}{2}\{\phi(\triangle + m^2)^{\frac{1}{2}}\phi + \dot{\phi}(\triangle + m^2)^{-\frac{1}{2}}\dot{\phi}\}(x)$$

with the function f on M. In particular, a_1 is the function which counts the number of particles present, and it commutes with the Hamiltonian H. The commutant $\mathcal{A}_h^0$ of a_1 is generated by the elements a_f and their time-derivatives $\dot{a}_f = \mathrm{i}[H, a_f]$. It is the usual quantum deformation of the algebra of functions on the tangent bundle $Y_M^{\text{part}} = TX_M^{\text{part}}$. It does not contain the operators ϕ_f and $\dot{\phi}_f$ which describe the actual classical fields: they do not commute with the number operator a_1, but are the analogues of the elements b, b^* in the usual Stone-von Neumann theorem. They are not part of the 'picture' in which one sees particles, though of course from a different point of view in different circumstances we might see the whole algebra[18] $\mathcal{A}_h$ as approximately commutative, giving us a picture of fields rather than of particles.

[18] Just like $\mathbb{C}\langle b, b^*\rangle$, the algebra $\mathcal{A}_h$ is $\mathbb{Z}$-graded, with the 'particle' algebra $\mathcal{A}_h^0$ as its degree 0 part.